Radiation Fundamentals II

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- Spectral lines of Astronomical interest
- Spectral line formation
- Polarization: application Faraday Rotation

Essential Radio Astronomy, Condon, J. & Ransom, S.
https://www.cv.nrao.edu/~sransom/web/xxx.html
Spectral Lines: HI 21cm line

- Study of neutral component of ISM -- cold and warm; our Galaxy and external galaxy (cold – 60 K, 30 cm$^{-3}$, 5x10$^{19}$ cm$^{-2}$; Warm – 8000 K, 0.2 cm$^{-3}$, 10$^{18}$ cm$^{-2}$)

- Kinematics of our Galaxy and external galaxy (eg. spiral galaxy)

- Large scale structure (nearby z<0.16; < 700 Mpc)
Spectral Lines: CO line

12CO, 13CO, C18O

Image of Orion region

12C16O1-0
115.27 GHz

Study of cold molecular gas in ISM;
(10 K, 10^3 cm^{-3}, 20 pc)

Proxy for H_2; cloud properties (need to know X_{co}).
Spectral Lines: Radio Recombination Lines

Recombination lines: recombined electrons making transition at high quantum states

Study ionized component of ISM
($T_e \sim 10^4$ K, $n_e \sim 0.1$ cm$^{-3}$ WIM; $\sim 100$ cm$^{-3}$ HII)
Spectral Lines: Astronomical Masers

Stimulated microwave spectral line emission

OH, H$_2$O, CH$_3$OH, SiO, HRRL

Structure of Milky Way

Plan view of Milky Way. Dots are the locations of newly formed OB-type stars determined from VLBI trigonometric parallaxes using associated H$_2$O or methanol maser emission. The Galactic center is denoted by the plus (+) sign at (0,0) kpc; the Sun is labeled in yellow at (0,8.4) kpc.
Spectral line formation

Optically thin case: no background radiation

\[ u_\nu = \frac{1}{4\pi} N_u A_{ul} \phi(\nu) h\nu_0 \text{ ergs/s/cm}^3/\text{Hz/strad} \]

\[ P_\nu = 4\pi \frac{1}{4\pi} N_u A_{ul} \phi(\nu) h\nu_0 \frac{4\pi}{3} R_s^3 \text{ ergs/s/Hz} \]

\[ S_L(\nu) = \frac{P_\nu}{4\pi R_d^2} \propto N_u A_{ul} 2R_s \left(\frac{R_s}{R_d}\right)^2 \phi(\nu) h\nu_0 \text{ ergs/s/cm}^2/\text{Hz} \]

\[ \propto \left( N_u 2R_s A_{ul} \phi(\nu) h\nu_0 \Omega_s \right) \text{ ergs/s/cm}^2/\text{Hz} \]

What we want?

- \( n_d \text{ (cm}^{-3}) \); \( N \sim n_d 2R_s \text{ (cm}^{-2}) \)
- \( T \text{ (K)}, R_s \text{ (pc)} \)

Prediction for obs

Emissivity

Spectral density radiated by the source

Flux density of the line

Column density of atoms in level u
Spectral line formation

Einstein $A$ coefficient (spontaneous emission coefficient)

Classical approximation: $A_{ul}$ is average spectral power emitted by a dipole divided by the photon energy.

HI 21cm line:

$$A_{F=10} = 2.85 \times 10^{-15} \text{ s}^{-1} \left(\sim 11 \text{ Myr half-life}\right)$$

$^{12}\text{C}^{16}\text{O}$ rotational trans:

$$A_{J=10} = 7.2 \times 10^{-8} \text{ s}^{-1} \left(\sim 0.4 \text{ yr; } \propto J^3\right)$$

RRL transitions:

$$A_{n+1,n} = 5.3 \times 10^9 \left(\frac{1}{n^5}\right) \text{ s}^{-1} \left(n=100; \sim 2 \text{ sec}\right)$$
Spectral Line formation

Energy level population

\[
\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_{ul}}{k_B T}} \quad T \text{ is a unique temperature in TD equilibrium}
\]

Otherwise \( T = T_{ex} \) (excitation temperature)

Statistical Equilibrium

For two level system

\[
N_u A_{ul} + N_u C_{ul} + N_u B_{ul} \frac{I_\nu}{c} = N_l C_{lu} + N_l B_{lu} \frac{I_\nu}{c}
\]

Multi-level system

\[
N_i \sum_j R_{ij} = \sum_j N_j R_{ji}
\]

Solution to this equation gives the level population; \( T_{ex} \) can be different for different levels
Spectral line formation

Energy level population: LTE approximation

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{\frac{E_{ul}}{kT}} \quad T = T_k \text{ the kinetic temperature}$$

(In general for LTE $T=T_{ex}$ is same of all level; first let us take $T=T_k$)

Condition: \[ C_{ul} \approx A_{ul} \]

\[ n^* \sigma v \approx A_{ul} \]

$n^*$ critical density (density of collision partner); \[ \sigma \text{ cross section; } v \text{ mean velocity} \]

HI 21 cm line: \[ n^* \ll 1 \text{ cm}^{-3} \]

$^{12}$C$^{16}$O J=1-0 line: \[ n^* \sim 10^3 \text{ cm}^{-3} \quad (T_k \sim 20 \text{ K}) \]

RRLs: \[ n^* \sim 90 \text{ cm}^{-3} \quad (T_k \sim 10^4 \text{ K}; n=100) \]
Spectral line formation

Energy level population: LTE approximation

\[
\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_u}{k_b T}} \quad T = T_k \text{ same for all levels}
\]

\[
\frac{N_u}{n_d} = \frac{g_u e^{-\frac{E_u}{k_b T_k}}}{Q}
\]

\[
Q = \sum_i g_i e^{-\frac{E_i}{k_b T_k}} \quad \text{Partition function}
\]

HI 21cm line

\[
Q \approx g_0 + g_1 e^{\frac{\hbar \nu \lambda_0}{k_b T_k}} \quad \frac{\hbar \nu \lambda_0}{k_b} \approx 0.08 \text{ K}
\]

\[
\approx 1 + 3 = 4 \quad (\text{This is true for other values of T encountered in the ISM})
\]

\[
N_u \approx \frac{3}{4} n_d
\]
Spectral line formation

Energy level population: LTE approximation

$^{12}\text{C}^{16}\text{O}$ J=1-0 line

$Q_{rot} \approx \frac{2k_BT_k}{h\nu_{10}} = 0.36T_k$

$N_u \approx n_d \times \frac{3 e^{-\frac{h\nu_{10}}{k_BT_k}}}{0.36T_k} \quad \text{for } \nu_{J=10} = 115.27 \text{ GHz transition}$

(see Mangum & Shirley 2015; Turner 1991)

Hydrogen RRLs

Number density of atoms in quantum state n

$N_n \propto n^2 \frac{n_p n_e}{T_e^{3/2}}$

$\propto n^2 \frac{n_e^2}{T_e^{3/2}} \quad n_d \approx n_e$

• Q has contribution from rotation, vibration, ...
• Rotation – lowest energy state
• Q ~ $Q_{rot}$
• $T_k \sim 20$ K

• Saha-Boltzmann equation in Q
• Q diverges for hydrogenic atoms
• For fulling ionized gas in ISM
  $n_d \sim n_p$ the proton density ~ $n_e$
  $T_k = T_e$ the electron temperature
Spectral line formation

Optically thin case; no background radiation

What we want?

\[ n_d \text{ (cm}^{-3}\text{); } N \sim n_d 2R_s \text{ (cm}^{-2}\text{)} \]

\[ T \text{ (K), } R_s \text{ (pc)} \]

Prediction for obs

Flux density of the line

\[ S_L(\nu) = \frac{P_\nu}{4\pi R_d^2} N_u A_{ul} 2R_s \left( \frac{R_s}{R_d} \right)^2 \phi(\nu) h\nu_0 \text{ ergs/s/cm}^2/\text{Hz} \]

\[ \propto N_u 2R_s A_{ul} \phi(\nu) h\nu_0 \Omega_s \text{ ergs/s/cm}^2/\text{Hz} \]

\[ N_u \text{ in terms of the total density} \]

Normalized Gaussian
Spectral line formation

Not optically thick; background radiation

\[ I_\nu(0) \quad \rightarrow \quad I_\nu(\tau_\nu) \]

\[ I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \text{ ergs/s/cm}^2/\text{Hz/strad} \]

\[ \tau_\nu = \int \kappa_\nu ds \approx \kappa_\nu \times 2R_s \quad \kappa_\nu \text{ absorption coefficient} \]

\[ S_\nu = \frac{u_\nu}{\kappa_\nu} \text{ ergs/s/cm}^2/\text{Hz/strad} \quad \text{(Source function)} \]
Spectral line formation

**Optically thin; no background case**

\[ I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \]

Line flux density

\[ S_L(\nu) = S_\nu \tau_\nu \Omega_s \quad \text{For} \quad I_\nu(0) = 0; \tau_\nu << 1 \]

\[ = u_\nu 2R_s \Omega_s \quad \text{ergs/s/Hz/cm}^2 \]
Spectral line formation

Source function

\[ S_\nu = B(T) \approx \frac{2k_bT}{\lambda^2} \]

In TD equilibrium by Kirchhoff’s law

LTE approximation

\[ \frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_u}{k_bT}} \quad T = T_{ex} \]

\[ S_\nu \approx B(T_k) \approx \frac{2k_bT_k}{\lambda^2} \quad T_{ex} = T_k \text{ same for all levels} \]

\[ S_\nu \approx B(T_{ex}) \approx \frac{2k_bT_{ex}}{\lambda^2} \quad T_{ex} \text{ same for all levels} \]
Spectral line formation

Solution to RT in temperature

\[ I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}} + S_{\nu}(1 - e^{-\tau_{\nu}}) \]

\[ S_{\nu} \approx B(T_{ex}) \approx \frac{2k_{b}T_{ex}}{\lambda^{2}} \quad T_{ex} \text{ same for all levels} \]

\[ I_{\nu}(0) = \frac{2k_{b}T_{\nu,b}}{\lambda^{2}} \quad T_{\nu,b} \text{ brightness temperature} \]

\[ T_{\nu}(\tau_{\nu}) = T_{\nu,b} e^{-\tau_{\nu}} + T_{ex}(1 - e^{-\tau_{\nu}}) \quad \text{K} \]
Spectral line formation

Line optical depth: LTE case

\[ \frac{dI_\nu}{ds} = -\kappa_\nu I_\nu \, ds \quad \text{Stimulated emission} \]

\[ \kappa_\nu = \left( \frac{\hbar \nu_0}{c} \right) \left( N_l \, B_{lu} - N_u \, B_{ul} \right) \phi(\nu) \]

\[ \tau_\nu = \int \kappa_\nu \, ds \propto N_u \, 2R_s \, A_{ul} \, \frac{1}{\nu_0 T_{ex}} \phi(\nu) \quad \text{For } \frac{\hbar \nu_0}{k_b T_{ex}} << 1 \]
Spectral line formation

Peak line optical depth: LTE case

\[ \tau_\nu \propto N_u \, 2R_s \, A_{ul} \, \frac{1}{\nu_0 T_{ex}} \, \phi(\nu) \]

\[ \phi(\nu_0) = \frac{0.94}{\Delta \nu} \quad \Delta \nu \text{ is FWHM} \]

\[ \frac{\Delta \nu}{\nu_0} = \frac{\Delta v}{c} \quad \Delta v \text{ is FWHM (usually) km s}^{-1} \]

Gaussian profile function

HI 21cm line

\[ \tau_\nu(\nu_0) = 5.1335 \times 10^{-19} \, \frac{N_H \text{ (cm}^{-2})}{T_s \text{ (K) } \Delta v \text{ (km s}^{-1})} \]

\[ N_H \text{ column density of HI, } T_{ex} = T_s \text{ spin temperature} \]

\[ N_u = \frac{3}{4} \, n_d; \, A_{ul} \text{ is const} \]

Freq = 1420.405 MHz
Spectral line formation

Peak line optical depth: LTE case

\[ \tau_\nu \propto N_u \ 2R_s \ A_{ul} \ \frac{1}{\nu_0 T_{ex}} \ \phi(\nu) \]

\[ \phi(\nu_0) = \frac{0.94}{\Delta\nu} \quad \Delta\nu \text{ is FWHM} \]

\[ \frac{\Delta\nu}{\nu_0} = \frac{\Delta v}{c} \quad \Delta v \text{ is FWHM (usually) km s}^{-1} \]

Gaussian profile function

CO J=1-0 transition

\[ \tau_{\nu_{1,0}}(\nu_0) = 3.95 \times 10^{-15} \left(1 - e^{-\frac{h\nu_{1,0}}{k_b T_{ex}}} \right) \frac{N_{CO}(\text{cm}^{-2})}{T_{ex}(\text{K}) \ \Delta v(\text{km s}^{-1})} \]

\[ N_{CO} \text{ column density of CO, } T_{ex} \text{ excitation temperature} \]

(did not use \( h\nu/k_b T_{ex} \ll 1 \) approximation; \( T_{ex} \sim 20 \text{ K} \)
Spectral line formation

Peak line optical depth: LTE case

\[ \tau_\nu \propto N_u \frac{2R_s}{\nu_0 T_{ex}} A_{ul} \frac{1}{\nu_0} \phi(\nu) \]

\[ \phi(\nu_0) = \frac{0.94}{\Delta \nu} \quad \Delta \nu \text{ is FWHM} \]

\[ \frac{\Delta \nu}{\nu_0} = \frac{\Delta v}{c} \quad \Delta v \text{ is FWHM (usually) km s}^{-1} \]

Gaussian profile function

RRL alpha transition

\[ \tau_\nu(\nu_0) = 1.92 \times 10^{3} \frac{EM \text{ (pc cm}^{-6})}{T_e^{5/2} \text{(K)} \Delta \nu \text{(KHz)}} \]

\[ EM = \int n_e^2 ds \quad \text{Emission measure} \]

For ionized gas: \( T_{ex} = T_k = T_e \)

\[ N_u \propto n^2 \quad 1/T_e^{3/2} \quad n_e^2 \]

\[ A_{ul} \propto 1/n^5 \]

RRL alpha freq \( \propto 1/n^3 \)

Knowing source function and optical depth we can get \( S_L \) from RT solution
Spectral line formation

**Excitation temperature** $T_{\text{ex}}$

Consider two level system

\[ N_u A_{ul} + N_u C_{ul} + N_u B_{ul} \frac{I_\nu}{c} = N_l C_{lu} + N_l B_{lu} \frac{I_\nu}{c} \]

Statistical equilibrium

**$T_{\text{ex}}$ definition**

\[ \frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_{ul}}{k_b T_{\text{ex}}}} = \frac{C_{lu} + B_{lu} I_\nu}{A_{ul} + C_{ul} + B_{ul} I_\nu} \]

- **Radiation dominates**
  \[ \frac{B_{lu} I_\nu}{A_{ul} + B_{ul} I_\nu} \]
  \[ T_{\text{ex}} \rightarrow T_b \]

- **Collision dominates**
  \[ \frac{C_{lu}}{C_{ul}} = \frac{g_u}{g_l} e^{-\frac{E_{ul}}{k_b T_k}} \]
  \[ T_{\text{ex}} \rightarrow T_k \]

**Critical density**

$C_{ul} \sim A_{ul}$
Spectral line formation

Non-LTE effects

\[ \frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_{ul}}{k_b T_{ex}}} \]

\[ = \frac{b_u g_u}{b_l g_l} e^{-\frac{E_{ul}}{k_b T_k}} \]

LTE level population with T=T_k

\[ b_u, b_l \text{ are departure coefficient} \]

Solve for \( b_u, b_l \) using statistical equilibrium equation.

Maser emission

\[ N_u > N_l \quad \text{Upper level population greater than lower level} \]

\[ \frac{N_u}{N_l} = \frac{g_u}{g_l} e^{\frac{E_{ul}}{k_b T_{ex}}} \]

\( T_{ex} \) is negative !!

\[ I_\nu(\tau_\nu) \approx I_\nu(0) e^{\tau_\nu} \quad \text{Optical depth is negative} \rightarrow \text{exponential amp.} \]
**Polarization**

**Polarization of galactic background emission**

![K, 23 GHz](image1)

![L, 1.4 GHz](image2)

**WMAP image of polarized galactic synchrotron emission at 3° scale**

**DRAO image of galactic polarized synchrotron Emission smoothed to 3° scale. The difference in emission structure is dominated by Faraday depolarization.**
Polarization

Measurement

General elliptical pol from two (orthogonal) linear pol

\[
\vec{E} = (E_x e^{i\phi_x} \hat{x} + E_y e^{i\phi_y} \hat{y}) e^{i\omega t}
\]

\(E_x, E_y, \delta = \phi_x - \phi_y\)

p, fractional pol

Stokes parameters (incoherent source)

\[
I = < E_x^2 > + < E_y^2 >
\]

\[
Q = < E_x^2 > - < E_y^2 >
\]

\[
U = 2 \text{ Re} < E_x E_y^* > = 2 < E_x E_y \cos(\delta) >
\]

\[
V = 2 \text{ Im} < E_x E_y^* > = 2 < E_x E_y \sin(\delta) >
\]
Polarization

**Measurement**

Stokes parameters (incoherent source)

\[
I = < E_x^2 > + < E_y^2 > \\
Q = < E_x^2 > - < E_y^2 > \\
U = < E_{45}^2 > - < E_{-45}^2 > \\
V = < E_l^2 > - < E_r^2 >
\]

\[
\chi = \frac{1}{2} \tan^{-1} \left( \frac{U}{Q} \right) \hspace{1cm} \text{Angle of linear polarization}
\]

\[
P = Q + iU = |P| \ e^{2i\chi} \hspace{1cm} \text{Linear polarization in complex notation; |P| lin. pol flux density}
\]

Hamaker & Bregman (1996)
Polarization

Measurement

General elliptical pol from two (orthogonal) circular pol
Polarization

Faraday Rotation

- Linear pol light → sum of right and left circular pol
- Right and left circular pol velocities are different (due to coupling with the cyclotron motion of thermal electrons)

\[ 2\Delta \psi = \left( \frac{2\pi}{\lambda_r} - \frac{2\pi}{\lambda_l} \right) \Delta s \]

\[ \Delta \psi \approx \lambda^2 (m^2) \times 0.8125 \int_{source} B_{||}(\mu G) n_e (cm^{-3}) ds (pc) \]

For \( \omega >> \omega_p \) & \( \omega >> \omega_c \)

\[ \approx \lambda^2 (m^2) RM \ (rad \ m^{-2}) \]

RM is the rotation measure

B \( \sim 1\mu G \ f_c \sim 2.8 \) Hz
\( n_e \sim 0.1 \ cm^{-3} \ f_p \sim 2.8 \) KHz
Polarization

RM measurement

\[ \Delta \psi \approx \lambda^2 (m^2) \text{RM} \ (\text{rad m}^{-2}) \]

\[ \text{RM} = \frac{d\chi}{d\lambda^2} \]

Angle of the measured linear pol

\[ P = Q + iU = |P| e^{2i\chi} \]

\[ |P| \] is the flux density of linear polarization

Combining RM and DM one could get \( B_\parallel \) field

(Haverkorn et al 2015)
Polarization

Faraday tomography or RM synthesis

Faraday depth (not RM; RM is the integral over the whole source)

\[ \phi(r) = 0.8125 \int_{r_{obs}}^{r} B_\parallel n_e ds \]

Change in pol angle due to Faraday depth

\[ \Delta \psi(r) = \lambda^2 \phi(r) \]

Fourier transform \((\Phi, \lambda^2)\)

\[ P_{obs}(\lambda^2) = \int P(r) e^{2i\phi(r)\lambda^2} d\phi(r) \]

Linear pol at r average over the source

\[ Q_{obs} + iU_{obs} = Q(r) + iU(r) \]
Thank You