Title: Correlation Function to Power Spectrum Transformations

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I. Introduction

The true power spectrum, \( P(f) \), is exactly and unambiguously given as the Fourier transform of the true autocorrelation function, \( R(\tau) \), which must be known for all \( \tau, -\infty \leq \tau \leq \infty \). However, when we step from the mathematical world to the real world the relation between \( N \) samples of an approximate spectrum, \( P^*(k\Delta f) \), \( 0 \leq k \leq N-1 \), and \( N \) samples of an autocorrelation function, \( R(n\Delta \tau) \), \( 0 \leq n < N-1 \), becomes somewhat arbitrary and ambiguous. (In further notation the *, \( \Delta f \), and \( \Delta \tau \) will be dropped to give \( P(k) \) and \( R(n) \) as the spectral estimate and sampled autocorrelation function.) One transformation gives one approximation to \( P(k) \) and a different transformation gives a different approximation; unless a criteria of "best" is chosen, the choice is arbitrary.

II. Transform Criteria

The criteria which will be used here to select an optimum transformation are the following:

1) As is widely discussed in the literature (see Blackman and Tukey [1], Weinreb [2], and Rabiner and Gold, p. 88 [3]), \( P(k) \) is a convolution of \( P(f) \) with an equivalent filter shape function, \( W(f-k\Delta f) \). We desire that \( W \) be as narrow and free of spurious lobes as possible. These two criteria conflict and the compromises are discussed in the literature. A usual procedure is to adopt a narrow \( W(f) \) by uniform weighting of \( R(n) \) and deal with spurious lobe reduction in later processing by combining adjacent spectral points; i.e., a new estimate \( P'(k) = aP(k-1) + bP(k) + aP(k+1) \) is formed where \( a \) and \( b \) are selected constants.

However, with little effect on the width and lobe suppression, it is possible to choose the transformation to meet other criteria given below.
2) Imperfections in the sampler tend to produce a large and somewhat
unstable spurious signal at zero frequency. This results from DC offsets and
leakage of the sampler clock signal or its harmonics into the sampler input. For
this reason, it is highly desirable to have spectral values $P(k)$ for $k \neq 0$ independent
of the zero frequency signal; i.e., the window function, $W(f-k\Delta f)$, should have a
zero at $f = 0$ for all $k$.

3) A convenient transformation is the Fast-Fourier-Transform, FFT, as
implemented with the Cooley-Tukey algorithm. The most widely available FFT
algorithms are for $N_D$ points equal to a power of 2. Digital correlators are
often built to also have a power of 2 number of channels. This is somewhat
unfortunate as criteria 2) is easily met with $N_D = 2(N-1)$ where $N$ is the number
of correlator channels; i.e., a correlator with a power of two channels plus one
would be convenient. However, a remedy exists which allows $N_D = 2N$.

4) The sampling theorem applied in the frequency domain determines the
maximum spacing of frequency points, $\Delta f = f_s/2(N-1)$, which will preserve all
information in the autocorrelation function which is band limited to $0 \leq \tau \leq
(N-1)/f_s$ where $f_s = 1/\Delta \tau$ is the sampling frequency. The required maximum angle
argument in the FFT is then $2\pi(k\Delta f)(n\Delta \tau) = 2\pi kn/2(N-1) = 2\pi kn/N_D$. Note that
$N_D = 2N$ provides sufficiently close frequency points, $N_D = N$ does not, and
$N_D = 2(N-1)$ is the minimum size transform. Also note that only $N$ input data points
are available for a transform having $N_D > N$; the remaining data points can either
be made zero or repeats of the first $N$ points.

III. Definition of Transforms

We will compare 3 possible transform equations in the light of the-above
criteria. The first of these, $P_1$, defined below is the most obvious choice if
criteria 2) is not considered:

\[ P_1(k) = 2 \sum_{n=0}^{2N-1} R(n) \cos(\frac{2\pi nk}{2N}) - R(o) \]

where \( k \) is an integer ranging from 0 to \( N-1 \) in all equations. Thus \( P_1(k) + R(o) \) is twice the real part of a 2N point DFT of the real function \( R(n) \). Since \( R(n) = 0 \) for \( n \geq N \), the upper limit in the summation could be \( N-1 \), but this would not be in the form of a standard DFT since the angle argument necessarily contains 2N.

An equivalent reflected version of this DFT can be written as:

\[ P_1(k) = \sum_{n=0}^{2N-1} R'(n) \cos(\frac{2\pi nk}{2N}) \]

where

- \( R'(n) = R(n) \quad 0 \leq n \leq N-1 \)
- \( R'(n) = 0 \quad n = N \)
- \( R'(n) = R(2N-n) \quad N+1 \leq n \leq 2N-1 \)

Another selection of transform is suggested by Blackman and Tukey [1, p. 35] and is given by

\[ P_2(k) = 2 \sum_{n=0}^{2N-3} R(n) \cos[\frac{2\pi nk}{(2N-2)}] - R(o) - R(N-1)\cos\pi k \]

where the substitution \( N-1 = m \) is made in the original notation and the summation is written in the form of a real part of a 2(N-1) point DFT of the real function \( R(n) \) which is 0 for \( n \geq N \). This transform meets criteria 2); \( P_2(k) = 0 \) for all \( k \) when \( R(n) \) is constant with \( n \) as is produced by a zero-frequency signal.

A third transform which is a 2N point DFT and meets criteria 2) can be obtained by adding an \((N+1)\)th point to \( R(n) \). This can be done with surprisingly little deleterious effects (see Figure 1) by defining \( R(N) = R(N-2) \) as the extra
The solid line in the above figure is the true spectrum consisting of a constant plus a 40% ripple at a frequency having a Fourier component at $R(N-2)$. The bottom curve with + symbols is the normal DFT, $P_1(f)$ and the top + curve is the modified DFT with an added term, $R(N) = R(N-2)$. There is surprisingly little difference between the curves. The points represented with filled squares are weighted versions of the transform and show large attenuation of the ripple term since it is close to the resolution limit of the system (i.e., a ripple at $R(n)$ for $n \geq N$ is totally ignored).
point, and then in analogy to \( P_2 \),

\[
P_3(k) = 2 \sum_{n=0}^{2N-1} R(n) \cos(2\pi nk/2N) - R(0) - R(N) \cos \pi k
\]
or the equivalent form,

\[
P_3(k) = 2 \sum_{n=0}^{N-1} R(n) \cos(2\pi nk/2N) - R(0) + R(N) \cos \pi k
\]

IV. Transform Properties and Weighting

Some of the properties of these three transforms are shown in Figure 2. Since \( P_2 \) requires a difficult transform, it will be dropped from further discussion.

It is also obvious from Figure 2 that weighting of the transform will be needed in most cases to reduce spurious lobes. The weighting affects the zero frequency response.

A unified method of describing weighting effects on both \( P_1 \) and \( P_3 \) can be obtained by considering a weighting function, \( w(n) \) which multiplies \( R(n) \) defined by two constants \( A \) and \( B \), and the equations

\[
w(n) = A + (1 - A) \cos(\pi n/N) \quad 0 \leq n \leq N - 1
\]

\[
w(N) = B
\]

The values of \( A \) and \( B \) for \( P_1 \) and \( P_3 \) and uniform, hanning, and Hamming weighting are given in Table I below:

| TABLE I. WEIGHTING FACTORS |
|-----------------------------|-----------------|-----------------|-----------------|
|                             | Uniform Weight | Hanning Weight  | Hamming Weight  |
| Normal DFT, \( P_1 \)       |                 |                 |                 |
| \( A \)                     | 0               | 0.500           | 0.540           |
| \( B \)                     | 0               | 0               | 0               |
| Modified DFT, \( P_3 \)     |                 |                 |                 |
| \( A \)                     | 1               | 0.500           | 0.540           |
| \( B \)                     | 0               | 0               | 0.0800          |
Fig. 2. Frequency response produced by the three transforms defined in the text, $P_1$, $P_2$, and $P_3$ are shown from top to bottom, respectively. The solid line shows the value of the transform point $P(15)$ for an $N = 32$ point autocorrelation function as the frequency of the correlated time function is varied from 0 to 1/2 the sampling frequency, $f_s/2$. The outputs of $P_k(14)$ and $P_k(16)$ are also shown with + and x symbols, respectively. The zero frequency response of $P_1(k)$ is $\frac{1}{32}$ of the peak for all $k \neq 0$ while $P_2(k)$ and $P_3(k)$ are exactly zero at zero frequency for all $k \neq 0$. 
In this formulation, where $R(N-2)$ is unweighted, $R(N) = R(N-2)$ for all cases, and the transform is defined as for $P_3$ in the previous section. $P_1$ is described by the same equations since $R(N)$ will be multiplied by $w(N) = B = 0$ and has no effect. For the case of $P_3$, $B$ is chosen to give zero response at zero frequency.

Note that hanning weighting gives zero DC response with the normal transform and $P_3$ will not give zero DC response unless $B = 0$; i.e., $P_3 = P_1$, in this case. For Hanning weighting the value of $B$ which nulls the DC response is 0.0800 as found by computer iteration for $N = 16, 32, \text{and } 64$. We then note that $B = 2A - 1$ to give zero DC response for all three weightings of the modified DFT. This relation has been checked for other values of $A$.

The response of a transform output point, $P_k(4)$, to input sinusoids of frequencies from zero to $f_s/2$ is shown in Figure 3 for the case of $N = 32$ and various transforms. The modified DFT for $A = 0.60, 0.65, \text{and } 0.70$ is shown in Figure 4.

A listing of the relevant part of a GWBASIC program used to evaluate transforms is shown in Figure 5 with arrows on key lines.

V. Conclusions

1) For transform convenience it is desirable to construct correlators with number of channels, $N$, equal to one plus a power of 2.

2) The hanning weighting is a good general purpose window for most radio astronomy observations. It gives zero DC response for any $N$ and has very low spurious lobes.

3) If the 65% increase in equivalent filter half-power width due to hanning is not tolerable, then the modified DFT, $P_3$, with zero DC response can be used.

4) Functions which give an intermediate trade-off of resolution vs spurious lobe level are the modified DFT with $A = 0.60, 0.65, \text{and } 0.70$. 

7
Fig. 3. Response of various transforms to input sinusoidal signals at frequencies from 0 to $f_s/2$. The unweighted transforms are shown at left, hanning weight of $P_1 = P_3$ is shown at top-right, and Hamming weight of $P_3$ is shown at bottom-right. The number of autocorrelation points is 32; i.e., lags from 0 to 31.
Fig. 4. Response of modified DFT, $P_3(4)$, to frequencies from 0 to $f_s/2$ for weighting factors 0.60, 0.65, and 0.70 which gives increasingly narrow resolution and higher spurious lobe level.
Fig. 5. GSBASIC program used to evaluate transforms. Printing and plotting subroutines are not shown.
REFERENCES

