National Radio Astronomy Observatory

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Spectral Density of Modes in a Rectangular Metal Box

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This note uses the approach given in [1] to calculate the spectral density of electromagnetic modes in a highly overmoded closed rectangular metal box of dimensions $L_x \times L_y \times L_z$.

A plane wave can be described by the wave equation $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp(i\, \mathbf{k.r} - i\omega t)$. The wavevector **k** has a magnitude equal to the wavenumber $k = 2\pi f/c = \omega/c$ radians/m, and its direction is that of propagation of the wave. **E** lies in a plane perpendicular to **k** and can be represented by two polarization components.

Within the box, each resonant mode can be represented as the superposition of two plane waves with the same wavenumber which interfere to satisfy the boundary condition $E_t = 0$ at the box walls. The field of the mode (n_x, n_y, n_z) inside the box is described by:

$$E_x = E_x(t).\cos(\pi n_x x/L_x).\sin(\pi n_y y/L_y).\sin(\pi n_z z/L_z)$$
$$E_y = E_y(t).\sin(\pi n_x x/L_x).\cos(\pi n_y y/L_y).\sin(\pi n_z z/L_z)$$
$$E_z = E_z(t).\sin(\pi n_x x/L_x).\sin(\pi n_y y/L_y).\cos(\pi n_z z/L_z),$$

Figure 1 shows the simple case of two plane waves in the plane of the page interfering to satisfy the boundary conditions at the walls of a long rectangular waveguide.

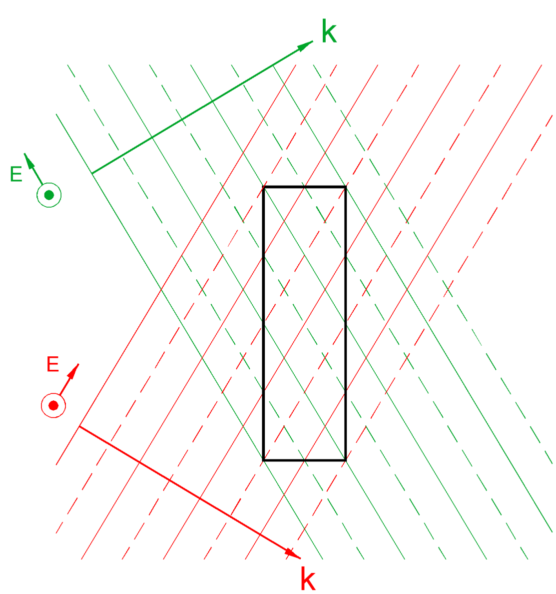

Fig. 1. Two plane waves of the same frequency interfering to satisfy the boundary conditions of a long metal waveguide. The solid lines indicate the E-field maxima and the dashed lines the minima at some instant of time. The wavevectors k are in the plane of the page, corresponding to the cutoff frequency of the 240 mode. Note that each wavevector is associated with two independent polarizations.

The phase of the components k_x, k_y, k_z, of **k** must vary along the sides of the box by an integral multiple of π radians; hence:

$$k_x = \pi n_x / L_x, \quad k_y = \pi n_y / L_y, \quad k_z = \pi n_z / L_z, \quad \text{where } n_x, n_y, n_z \text{ are integers.}$$

The resonant wavenumbers of the box are then

$$k = [(\pi n_x / L_x)^2 + (\pi n_y / L_y)^2 + (\pi n_z / L_z)^2]^{1/2} \text{ radians/m,}$$

which corresponds to resonant frequencies

$$f = kc/2\pi = (c/2)\,[(n_x/L_x)^2 + (n_y/L_y)^2 + (n_z/L_z)^2]^{1/2} \text{ Hz.}$$

In k-space, each resonant wavevector is represented by one point of a rectangular array of points, as shown in Fig. 2.

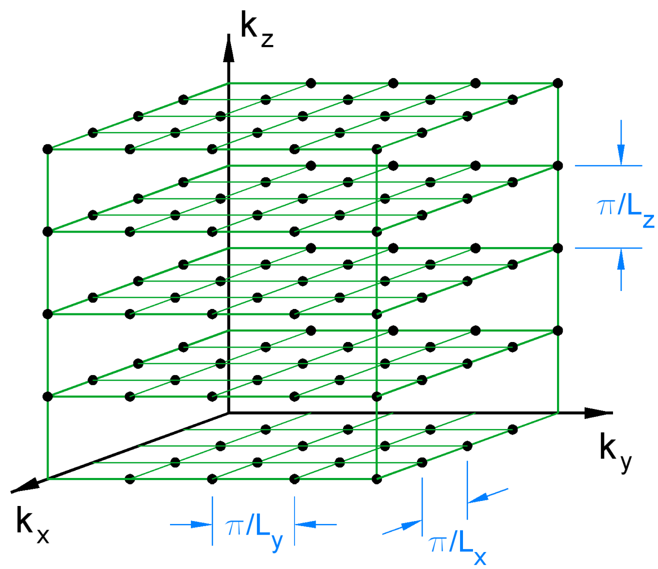

Fig. 2. Representation of the resonant wavevectors of a metal box of dimensions $L_x \times L_y \times L_z$ by points in k-space.

In the frequency range from f to f + df, the number of resonant wavevectors is given by the number of points in the k-space diagram for which the wavenumber k lies between k and k + dk. The volume of this octant shell is $\pi k^2 dk/2$. Since each point occupies a volume $(\pi / L_x) \times (\pi / L_y) \times (\pi / L_z)$, the density of points in k-space is $L_x L_y L_z / \pi^3$. The number of points between k and k + dk is $L_x L_y L_z k^2 dk / 2\pi^2$.

For each point in k-space, there are two independent polarizations (see Fig. 1) representing a pair of resonant modes at the same frequency. Since the wavenumber $k = 2\pi f/c$, the number of modes between f and (f + df) is therefore $2 \times L_x L_y L_z (2\pi f/c)^2 2\pi df / 2c\pi^2 = 8\pi L_x L_y L_z f^2 df/c^3$, which includes both polarization modes for each resonant wavevector. The spectral density of resonant modes is therefore $8L_x L_y L_z f^2/c^3$ modes/Hz.

Example

For a rectangular metal cavity 0.5 m × 0.5 m × 0.5 m at 30 GHz the spectral density of resonant modes $8\pi L_x L_y L_z f^2/c^3 = 8\pi(0.5)^3(30 \times 10^9)^2/(3 \times 10^8)^3 = 105 \times 10^{-6}$ modes/Hz. The average spacing between modes is therefore 9.5 kHz at 30 GHz.

Reference

[1] R. Loudon, "The Quantum Theory of Light," Oxford: The Clarendon Press, 1973.