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Title: Correlation Function to Power Spectrum Transformations

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Correlation Function to Power Spectrum Transformations

S. Weinreb

I. Introduction

The true power spectrum, $P(f)$, is exactly and unambiguously given as the Fourier transform of the true autocorrelation function, $R(\tau)$, which must be known for all τ, $-\infty \leqq \tau \leqq \infty$. However, when we step from the mathematical world to the real world the relation between N samples of an approximate spectrum, $P^*(k\Delta f)$, $0 \leqq k \leqq N-1$, and N samples of an autocorrelation function, $R(n\Delta\tau)$, $0 \leqq n < N-1$, becomes somewhat arbitrary and ambiguous. (In further notation the $*$, Δf and $\Delta\tau$ will be dropped to give $P(k)$ and $R(n)$ as the spectral estimate and sampled autorcorrelation function.) One transformation gives one approximation to $P(k)$ and a different transformation gives a different approximation; unless a criteria of "best" is chosen, the choice is arbitrary.

II. Transform Criteria

The criteria which will be used here to select an optimum transformation are the following:

1) As is widely discussed in the literature (see Blackman and Tukey [1], Weinreb [2], and Rabiner and Gold, p. 88 [3]), $P(k)$ is a convolution of $P(f)$ with an equivalent filter shape function, $W(f-k\Delta f)$. We desire that W be as narrow and free of spurious lobes as possible. These two criteria conflict and the compromises are discussed in the literature. A usual procedure is to adopt a narrow $W(f)$ by uniform weighting of $R(n)$ and deal with spurious lobe reduction in later processing by combining adjacent spectral points; i.e., a new estimate $P'(k) = aP(k-1) + bP(k) + aP(k+1)$ is formed where a and b are selected constants.

However, with little effect on the width and lobe suppression, it is possible to choose the transformation to meet other criteria given below.

2) Imperfections in the sampler tend to produce a large and somewhat unstable spurious signal at zero frequency. This results from DC offsets and leakage of the sampler clock signal or its harmonics into the sampler input. For this reason, it is highly desirable to have spectral values $P(k)$ for $k \neq 0$ independent of the zero frequency signal; i.e., the window function, $W(f-k\Delta f)$, should have a zero at $f = 0$ for all k.

3) A convenient transformation is the Fast-Fourier-Transform, FFT, as implemented with the Cooley-Tukey algorithm. The most widely available FFT algorithms are for ND points equal to a power of 2. Digital correlators are often built to also have a power of 2 number of channels. This is somewhat unfortunate as criteria 2) is easily met with $ND = 2(N-1)$ where N is the number of correlator channels; i.e., a correlator with a power of two channels plus one would be convenient. However, a remedy exists which allows $ND = 2N$.

4) The sampling theorem applied in the frequency domain determines the maximum spacing of frequency points, $\Delta f = f_s/2(N-1)$, which will preserve all information in the autocorrelation function which is band limited to $0 \leqq \tau \leqq (N-1)/f_s$ where $f_s = 1/\Delta\tau$ is the sampling frequency. The required maximum angle argument in the FFT is then $2\pi(k\Delta fk)(n\Delta\tau) = 2\pi kn/2(N-1) = 2\pi kn/ND$. Note that $ND = 2N$ provides sufficiently close frequency points, $ND = N$ does not, and $ND = 2(N-1)$ is the minimum size transform. Also note that only N input data points are available for a transform having $ND > N$; the remaining data points can either be made zero or repeats of the first N points.

III. Definition of Transforms

We will compare 3 possible transform equations in the light of the above criteria. The first of these, P_1, defined below is the most obvious choice if criteria 2) is not considered:

$$P_1(k) = 2 \sum_{n=o}^{2N-1} R(n)\cos(2\pi nk/2N) - R(o)$$

where k is an integer ranging from 0 to N-1 in all equations. Thus $P_1(k) + R(o)$ is twice the real part of a 2N point DFT of the real function R(n). Since R(n) = 0 for $n \geqq N$, the upper limit in the summation could be N-1, but this would not be in the form of a standard DFT since the angle argument necessarily contains 2N. An equivalent reflected version of this DFT can be written as:

$$P_1(k) = \sum_{n=0}^{2N-1} R'(n)\cos(2\pi nk/2N)$$

where

$$\begin{aligned} R'(n) &= R(n) && 0 \leq n \leq N-1 \\ R'(n) &= 0 && n = N \\ R'(n) &= R(2N-n) && N+1 \leq n \leq 2N-1 \end{aligned}$$

Another selection of transform is suggested by Blackman and Tukey [1, p. 35] and is given by

$$P_2(k) = 2 \sum_{n=0}^{2N-3} R(n)\cos[2\pi nk/(2N-2)] - R(o) - R(N-1)\cos\pi k$$

where the substitution $N-1 \equiv m$ is made in the original notation and the summation is written in the form of a real part of a 2(N-1) point DFT of the real function R(n) which is 0 for $n \geq N$. This transform meets criteria 2); $P_2(k) = 0$ for all k when R(n) is constant with n as is produced by a zero-frequency signal.

A third transform which is a 2N point DFT and meets criteria 2) can be obtained by adding an (N+1)th point to R(n). This can be done with surprisingly little deleterious effects (see Figure 1) by defining R(N) = R(N-2) as the extra

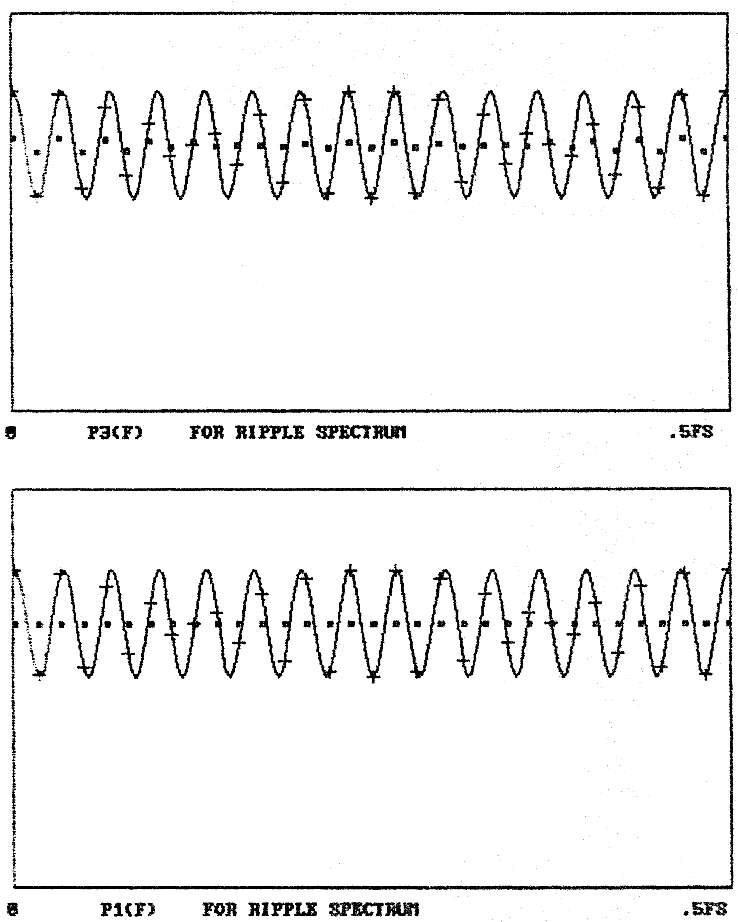

Fig. 1. The solid line in the above figure is the true spectrum consisting of a constant plus a 40% ripple at a frequency having a Fourier component at R(N-2). The bottom curve with + symbols is the normal DFT, $P_1(f)$ and the top + curve is the modified DFT with an added term, R(N) = R(N-2). There is surprisingly little difference between the curves. The points represented with filled squares are weighted versions of the transform and show large attenuation of the ripple term since it is close to the resolution limit of the system (i.e., a ripple at R(n) for $n \geq N$ is totally ignored).

point, and then in analogy to P_2,

$$P_3(k) = 2 \sum_{n=0}^{2N-1} R(n)\cos(2\pi nk/2N) - R(o) - R(N)\cos\pi k$$

or the equivalent form,

$$P_3(k) = 2 \sum_{n=o}^{N-1} R(n)\cos(2\pi nk/2N) - R(o) + R(N)\cos\pi k$$

IV. Transform Properties and Weighting

Some of the properties of these three transforms are shown in Figure 2. Since P_2 requires a difficult transform, it will be dropped from further discussion. It is also obvious from Figure 2 that weighting of the transform will be needed in most cases to reduce spurious lobes. The weighting affects the zero frequency response.

A unified method of describing weighting effects on both P_1 and P_3 can be obtained by considering a weighting function, w(n) which multiplies R(n) defined by two constants A and B, and the equations

$$w(n) = A + (1 - A)\cos(\pi n/N) \qquad 0 \leq n \leq N - 1$$

$$w(N) = B$$

The values of A and B for P_1 and P_3 and uniform, hanning, and Hamming weighting are given in Table I below:

TABLE I. WEIGHTING FACTORS

	Uniform Weight	Hanning Weight	Hamming Weight
Normal DFT, P_1			
A	0	0.500	0.540
B	0	0	0
Modified DFT, P_3			
A	1	0.500	0.540
B	0	0	0.0800

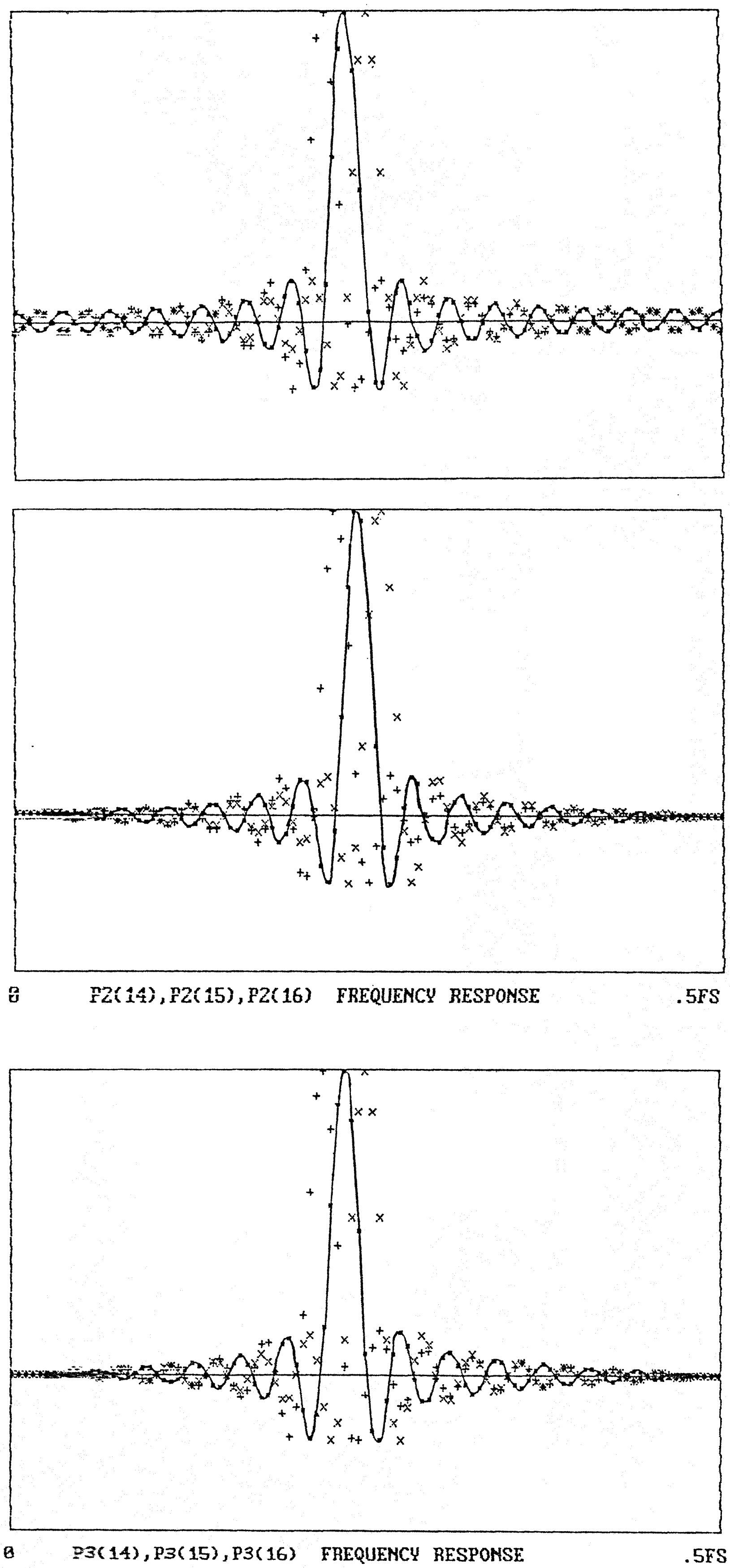

Fig. 2. Frequency response produced by the three transforms defined in the text, P_1, P_2, and P_3 are shown from top to bottom, respectively. The solid line shows the value of the transform point P (15) for an N = 32 point autocorrelation function as the frequency of the correlated time function is varied from 0 to 1/2 the sampling frequency, $f_s/2$. The outputs of $P_k(14)$ and $P_k(16)$ are also shown with + and x symbols, respectively. The zero frequency response of $P_1(k)$ is $\simeq$ 1/32 of the peak for all $k \neq 0$ while $P_2(k)$ and $P_3(k)$ are exactly zero at zero frequency for all $k \neq 0$.

In this formulation, where R(N-2) is unweighted, R(N) ≡ R(N-2) for all cases, and the transform is defined as for P_3 in the previous section. P_1 is described by the same equations since R(N) will be multiplied by w(N) = B = 0 and has no effect. For the case of P_3, B is chosen to to give zero response at zero frequency.

Note that hanning weighting gives zero DC response with the normal transform and P_3 will not give zero DC response unless B = 0; i.e., $P_3 \equiv P_1$, in this case. For Hamming weighting the value of B which nulls the DC response is 0.0800 as found by computer interation for N = 16, 32, and 64. We then note that B = 2A - 1 to give zero DC response for all three weightings of the modified DFT! This relation has been checked for other values of A.

The response of a transform output point, $P_k(4)$, to input sinusoids of frequencies from zero to $f_s/2$ is shown in Figure 3 for the case of N = 32 and various transforms. The modified DFT for A = 0.60, 0.65, and 0.70 is shown in Figure 4.

A listing of the relevant part of a GWBASIC program used to evaluate transforms is shown in Figure 5 with arrows on key lines.

V. Conclusions

1) For transform convenience it is desirable to construct correlators with number of channels, N, equal to one plus a power of 2.

2) The hanning weighting is a good general purpose window for most radio astronomy observations. It gives zero DC response for any N and has very low spurious lobes.

3) If the 65% increase in equivalent filter half-power width due to hanning is not tolerable, then the modified DFT, P_3, with zero DC response can be used.

4) Functions which give an intermediate trade-off of resolution vs spurious lobe level are the modified DFT with A = 0.60, 0.65, and 0.70.

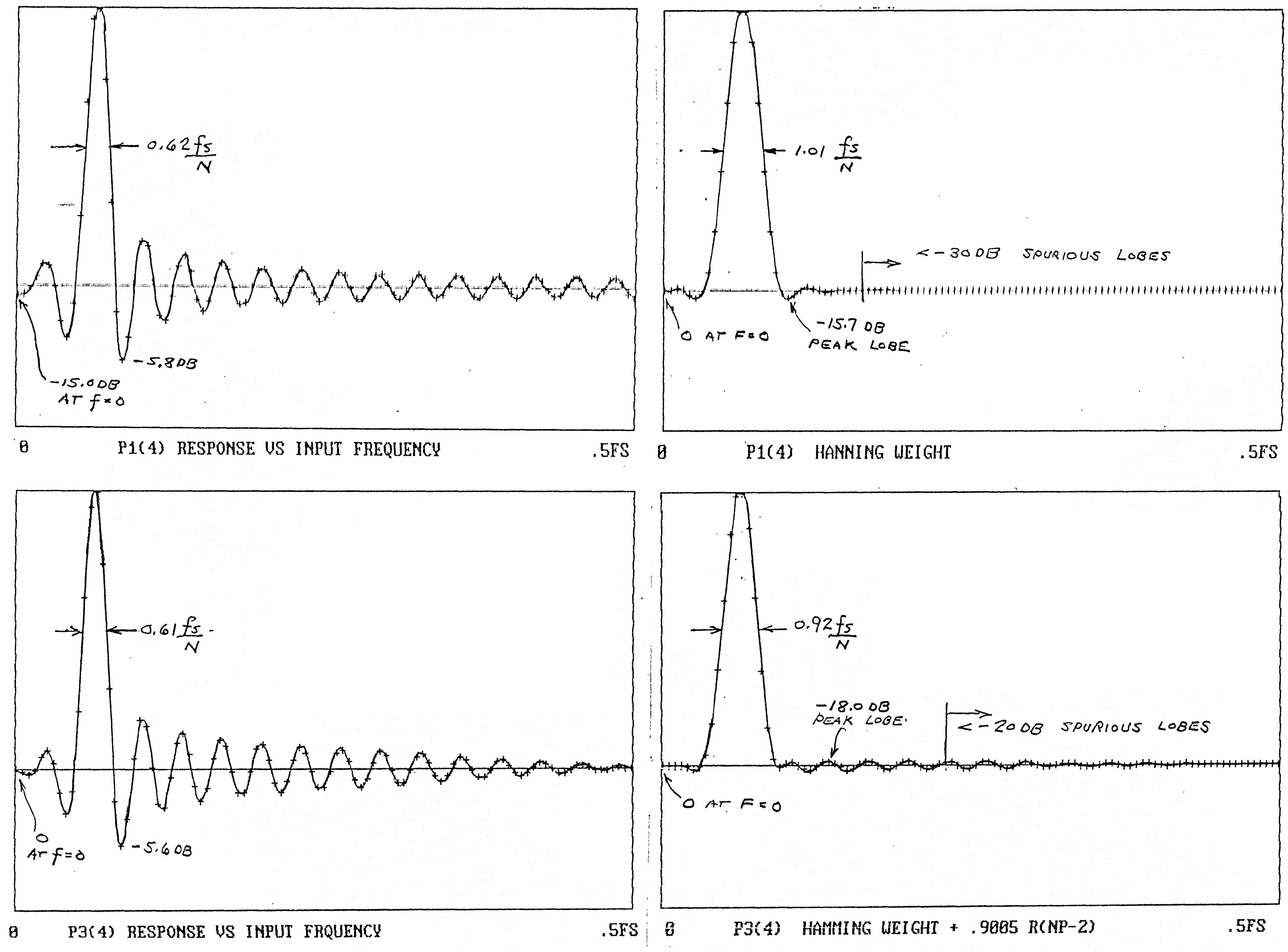

Fig. 3. Response of various transforms to input sinusoidal signals at frequencies from 0 to $f_s/2$. The unweighted transforms are shown at left, hanning weight of $P_1 = P_3$ is shown at top-right, and Hamming weight of P_3 is shown at bottom-right. The number of autocorrelation points is 32; i.e., lags from 0 to 31.

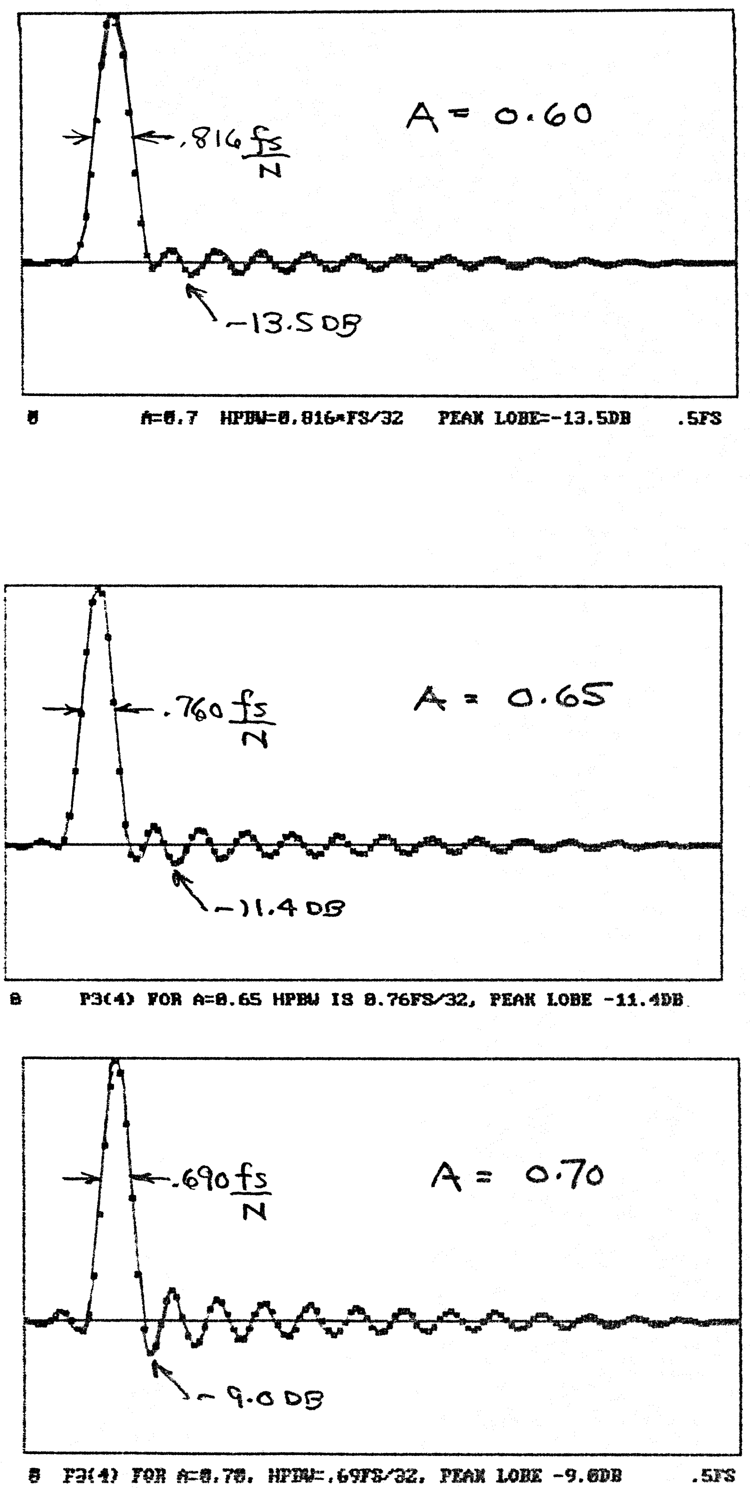

Fig. 4. Response of modified DFT, $P_3(4)$, to frequencies from 0 to $f_s/2$ for weighting factors 0.60, 0.65, and 0.70 which gives increasingly narrow resolution and higher spurious lobe level.

```
100 REM DFTABC 2/21/86
110 DIM R(128),P(4,128),RW(128)
120 LPRINT CHR$(27);"L008";CHR$(27);"E"; :REM LEFT MARGIN AND ELITE
130 PI=3.14159265#
140 SCREEN 0,1: COLOR 12,9 :CLS
480 PRINT "   K          F        P(1,F)     P(2,F)"
490 VIEW PRINT 2 TO 24
500 REM MAIN PROGRAM**********************************************************
505 A(1)=.65:A(2)=.75
510 K1=4:NP=32 :F1=0:F2=.5:JF=128
512 DF=(F2-F1)/JF
515 FOR J=0 TO JF:F=F1+J*DF
520 GOSUB 1000: REM GEN R(N) FOR F
530 FOR L=1 TO 2 :AW=A(L)
550 GOSUB 1500: REM WEIGHTED DFT
555 NEXT L
560 NEXT J
580 GOSUB 2000: REM PRINT TABLE
600 INPUT "SELECT K=1,2,OR 3 FOR PLOT OR K=0 TO HALT";K
610 IF K=0 THEN LIST 500-700
620 IF K=1 THEN GOSUB 2200
630 IF K=2 THEN GOSUB 2220
635 IF K=3 THEN GOSUB 2300
640 GOTO 600
999 END
1000 REM GENERATE R(N) FOR NORMALIZED FREQUENCY F *****************************
1020 B=2*PI*F
1030 FOR N=0 TO NP-1:R(N)=COS(B*N):NEXT
1040 R(NP)=R(NP-2)
1099 RETURN
1100 REM REFLECT R(N) *********************************************************
1120 R(NP)=0
1130 FOR N=NP+1 TO 2*NP-1: R(N)=R(2*NP-N): NEXT
1199 RETURN
1200 REM GENERATE R(N) FOR WHITE NOISE ****************************************
1210 R(0)=1
1220 FOR N=1 TO NP-1: R(N)=0: NEXT
1225 R(NP)=R(NP-2)
1230 RETURN
1300 REM WEIGHT R(N)**********************************************************
1305 B=PI/NP
1310 FOR N=0 TO NP
1320 RW(N)=R(N)*(AW+(1-AW)*COS(B*N))
1330 NEXT
1349 RETURN
1500 REM 2*NP TRANSFORM FOR P1 AND P3 *****************************************
1505 GOSUB 1300: REM WEIGHT
1510  A=2*PI/(2*NP)
1520 K=K1
1530 SUM=0 :AK=A*K
1540 FOR N=0 TO NP
1550 SUM=SUM+RW(N)*COS(AK*N)
1560 NEXT N
1575 P(L,J)=2*SUM -RW(0)-RW(NP)*COS(PI*K)
1578 IF L=2 THEN PRINT USING "##.####  ";K,F,P(1,J),P(2,J)
1580 IF P(L,J)>PMAX(L) THEN PMAX(L)=P(L,J)
1585 IF J=JF THEN FOR JK=0 TO JF:P(L,JK)=P(L,JK)/PMAX(L):NEXT JK
1599 RETURN
```

Fig. 5. GBBASIC program used to evaluate transforms.
Printing and plotting subroutines are not shown.

REFERENCES

[1] R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra, Dover, New York, 1958.

[2] S. Weinreb, "A Digital Spectral Analysis Technique and Its Application to Radio Astronomy," Technical Report #412, Research Lab of Electronics, M.I.T., Cambridge, August 30, 1963.

[3] L. R. Rabiner and B. Gold, Theory and Application of Digital Signal Processing, Prentice Hall, Englewood, NJ, 1975.